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The Existence in T-Spaces of Functions with Prescribed Alternations

B. GOPINATH AND R. P. KURSHAN

Bell Laboratories, Murray Hill, New Jersey 07974

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A $(k + 1)$-dimensional real vector space U of real-valued functions defined on a subset T of the real line is a Tchebycheff space (the linear space generated by a Tchebycheff system) iff the number of zeros and the number of alternations in sign of each nonzero element of U is at most k. We prove here that a necessary and sufficient condition that U be a Tchebycheff space is that for any $n \leqslant k$ (not necessarily distinct) points in T, there exists an element of U with exactly these points as zeros (except for possibly $k - n$ additional zeros), which alternates in sign across each zero. Furthermore, it is proved that if U is a Tchebycheff space of bounded functions, then the prescribed zeros can include points in the closure of T, if for these points "is equal to zero at" is understood to mean "is asymptotically zero at."

INTRODUCTION

Let $\mathscr{F}(T)$ denote the set of real-valued functions defined on a subset T of the real line $\mathbb{R}$. A $(k + 1)$-dimensional vector space $U \subset \mathscr{F}(T)$ over $\mathbb{R}$ is a Tchebycheff space (T-space) iff the number of zeros and the number of alternations in sign of each nonzero element of U is at most k. Various other characterizations of T-spaces can be found in [1].

When U is the T-space with basis $u_i(t) = t^i$, $i = 0,..., k$, defined on some interval $[a, b]$, the following is true. Suppose $z_{-1}, z_0, z_1,..., z_m$, with $z_{-1} = -\infty$, satisfy

$$z_i \in [a, b], \qquad i = 0, 1,..., m. \tag{1}$$

$$z_{i-1} \leqslant z_i, \qquad i = 1, 2,..., m. \tag{2}$$

$$-1 \leqslant m < k. \tag{3}$$

Then there exists a $\phi \in U$ such that

$$\|\phi\| = 1. \tag{4}$$

$$\phi(z_i) = 0, \qquad i = 0, 1,..., m. \tag{5}$$

$$t \in]z_{j-1}, z_j[\Rightarrow (-1)^j \phi(t) < 0, \qquad j = 0, 1,..., (m + 1) \tag{6}$$

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(with $z_{m+1} = +\infty$). Of course ϕ is the polynomial (appropriately normalized) with exactly the $m+1$ prescribed zeros $z_0, \dots, z_m$.

This result was stated by Krein for T-spaces of continuous functions on a closed interval [3, Lemma 3.4, p. 43] but was proved only for $m = k - 1$ (or when consecutive pairs of z_i's coincided); this result playing a fundamental role in his beautiful study of the geometry of moment spaces. However, for arbitrary T-spaces the former result (general m) is not exactly true. In order to be true the strict inequality in (6) must be replaced by $(-1)^i \phi(t) \geq 0$, even for T-spaces of continuous functions defined on a closed interval. Indeed, the example given by Zielke [4] of a T-space of degree 2 generated by 1, $t \sin t$, $t \cos t$ on $[0, \pi]$ has the property that any element with a zero at 0 must also have another zero.

However, if $m = k - 1$ (i.e., the number of prescribed zeros is k) and the prescribed zeros are distinct, then the strict inequality in (6) does obtain, inasmuch as an element of a T-space of degree k can have no more than k distinct zeros.

We prove here the corrected version of Krein's Lemma 3.4 (replacing $>$ by $\geq$ in (6)) and our proof is vaild for T-spaces of arbitrary functions defined on an arbitrary subset of $\mathbb{R}$. This is the content of Lemma 2.3 below.

Conversely, we prove that a sufficient condition for a $(k+1)$-dimensional linear subspace $U \subset \mathscr{F}(T)$ to be a T-space is that whenever the inequality (2) is strict and $m = k - 1$ then (5) and (6) obtain. Hence, the existence of such a ϕ having k prescribed zeros and the alternation property (6) gives another characterization of T-spaces:

THEOREM. *A $(k+1)$-dimensional real linear subspace $U \subset \mathscr{F}(T)$ is a T-space if and only if for each set of k distinct points in T there is a $\phi \in U$ such that ϕ has precisely these points as zeros, and alternates in sign across each zero.*

This is proved in Section 1 below.

In general when T is not closed or the elements of U are not continuous, it is useful to generalize the concept of a zero of a function to include points at which the function is asymptotically zero (this is needed, for example, to prove the oscillation theorem in [2]). We define a point $t \in \mathbb{R}$ to be an *asymptotic zero* of $u \in \mathscr{F}(T)$ if there exists a sequence $(t_n) \subset T$ such that

$$\lim_n t_n = t \qquad \text{and} \qquad \lim_n u(t_n) = 0.$$

When $U \subset \mathscr{B}(T)$, the set of bounded functions on T, we prove that the Lemma 2.3 below referred to previously remains true if any of the prescribed zeros are replaced by prescribed asymptotic zeros. In fact, for the prescribed asymptotic zeros, the associated sequence (t_n) may also be prescribed, up to a subsequence thereof. This is the content of Theorem 2.4 below.

This theorem is proved in four steps. When the number of prescribed zeros is k (the degree of the T-space concerned) it is simple to construct a ϕ satisfying (4)–(6). In Lemma 2.1 we show that this obtains for asymptotic zeros as well, except that the inequality in (6) may not be strict.

In Lemma 2.2, we prove the fundamental theorem for the case when the z_j's are distinct zeros, but may number fewer than k. In this case we find ϕ to be a uniform limit of polynomials each having k zeros. We pick the extra zeros in such a way that in the limit these zeros do not give rise to extra alternations in sign.

In Lemma 2.3 we show that a ϕ can be chosen which has an actual zero at as many as k (not necessarily distinct) points (thus allowing for "double zeros").

Finally, in Theorem 2.4 we add to Lemma 2.3 the possibility of asymptotic zeros for which, in addition, the associated sequence (t_n) may be prescribed (up to a subsequence thereof).

In Section 2 we require that the elements of U be bounded in order that U have the topology of uniform convergence induced by the sup norm $\|\cdot\|$ (rather than merely the topology of pointwise convergence induced by the ℓ_2-norm $\|\cdot\|_2$). Of course, on any finite-dimensional subspace of $\mathscr{B}(T)$, the sup norm and the ℓ_2 norm induce the same topology.

We denote the set of positive integers by $\mathbb{N}$.

1. A Characterization

THEOREM. *For some $T \subset \mathbb{R}$ suppose $U \subset \mathscr{F}(T)$ is a $(k+1)$-dimensional real vector space. Then U is a T-space of degree k iff the following holds: For every z_i, $i = -1, 0, 1, \ldots, m$ (with $z_{-1} = -\infty$) if*

$$z_i \in T, \qquad i = 0, 1, \ldots, m, \tag{1.1}$$

$$z_{i-1} < z_i, \qquad i = 1, 2, \ldots, m, \tag{1.2}$$

$$m = k - 1, \tag{1.3}$$

then there exists a $\phi \in U$ such that

$$\|\phi\|_2 = 1, \tag{1.4}$$

$$\phi(z_i) = 0, \qquad i = 0, 1, \ldots, m, \tag{1.5}$$

$$t \in \,]z_{j-1}, z_j[\, \cap T \Rightarrow (-1)^j \phi(t) < 0, \qquad j = 0, 1, \ldots, m+1 \tag{1.6}$$

with $z_{m+1} = +\infty$.

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) It can be easily shown that for every nonzero $u \in U$, the number of distinct zeros of u: $Z(u) \leq k$. Next, we show that any $u \in U$ which has exactly k zeros cannot have any alternations in sign between two consecutive zeros. Suppose $\psi \in U$ such that $\psi(z_i) = 0$, $i = 0, 1, \dots, k-1$ and $z_0 < z_1 < \cdots < z_{k-1}$. Furthermore suppose there exists t, t' such that $t, t' \in \,]z_{j-1}, z_j[\, \cap T$ and $\psi(t)\,\psi(t') < 0$. Now let ϕ be an element of U with zeros precisely at the points z_i, $i = 0, 1, \dots, k-1$. But from (1.6) there exists an α such that $\alpha\phi + \psi$ has in addition to the zeros z_i, $i = 0, 1, \dots, k-1$ a zero at t or t', which contradicts the fact that $Z(u) \leq k$ for every $0 \neq u \in U$.

Thus, (see [1, (4.12)]) the indicator function

$$N(u) = \begin{cases} Z(u) & \text{if } S^0(u) = 0 \\ Z(u) + 1 & \text{if } S^0(u) \neq 0 \end{cases}$$

satisfies $N(u) \leq k$ for all nonzero $u \in U$, whence U is a T-space (of degree k).

2. The Existence of Functions With Prescribed Alternations

(2.1) Lemma. *For some $T \subset \mathbb{R}$ suppose $U \subset \mathscr{B}(T)$ is a T-space of degree k. Let z_{jn}, $j = 0, 1, \dots, m$, $n \in \mathbb{N}$, and z_j, $j = -1, 0, 1, \dots, m$ (with $z_{-1} = -\infty$) satisfy*:

$$\forall n \in \mathbb{N},\ z_{jn} \in T,\ z_j \in cl\ T \quad and \quad \lim_n z_{jn} = z_j, \qquad j = 0, 1, \dots, m. \tag{2.1.1}$$

$$\forall n \in \mathbb{N}, \quad z_{(j-1)n} < z_{jn}, \qquad j = 1, 2, \dots, m. \tag{2.1.2}$$

$$m \leq k - 1. \tag{2.1.3}$$

Then there exists a $\phi \in U$ such that

$$\|\phi\| = 1. \tag{2.1.4}$$

$$\exists n_i,\ i \in \mathbb{N} \text{ such that } \lim_i \phi(z_{jn_i}) = 0 \quad j = 0, 1, \dots, m. \tag{2.1.5}$$

$$t \in \,]z_{j-1}, z_j[\, \cap T \Rightarrow (-1)^j \phi(t) \geq 0; \quad j = 0, 1, \dots, m+1 \text{ with } z_{m+1} = \infty. \tag{2.1.6}$$

Proof. From Section 1 it follows that for each n there exists a $\phi_n \in U$ such that

(a) $\|\phi_n\| = 1$.

(b) $\phi_n(z_{jn}) = 0$ *for* $j = 0, 1, \dots, m$, $n \in \mathbb{N}$.

(c) $t \in \,]z_{(j-1)n}, z_{jn}[\, \cap T \Rightarrow (-1)^j \phi_n(t) \geq 0$, $j = 0, 1, \dots, m+1$ *with* $z_{-1n} = -\infty$, $z_{(m+1)n} = \infty$.

Since U is a finite-dimensional real vector space

$$\Phi = \{u \in U \mid \| u \| = 1\}$$

is compact. Therefore since $\phi_n \in \Phi$ for all $n \in \mathbb{N}$, there exists a subsequence ϕ_{n_i}, $i \in \mathbb{N}$, and a $\phi \in \Phi$ such that

(d) $\| \phi_{n_i} - \phi \| < (1/i)$, $i \in \mathbb{N}$ (since $\| \cdot \|$ and $\| \cdot \|_2$ are equivalent on U).

That ϕ satisfies (2.1.4)–(2.1.6) is an elementary calculation.

(2.2) LEMMA. *For some $T \subset \mathbb{R}$, suppose $U \subset \mathscr{B}(T)$ is a T-space of degree k. Let z_j, $j = -1, 0, 1,..., m$ (with $z_{-1} = -\infty$) satisfy*

$$z_j \in T, \qquad j = 0, 1,..., m. \tag{2.2.1}$$

$$z_{j-1} < z_j, \qquad j = 1, 2,..., m. \tag{2.2.2}$$

$$-1 \leqslant m < k. \tag{2.2.3}$$

Then there exists a $\phi \in U$ such that

$$\| \phi \| = 1. \tag{2.2.4}$$

$$\phi(z_j) = 0 \qquad j = 0, 1,..., m. \tag{2.2.5}$$

$$t \in [z_{j-1}, z_j] \cap T \Rightarrow (-1)^j \phi(t) \leqslant 0; \qquad j = 0, 1,..., m+1 \tag{2.2.6}$$

with $z_{m+1} = +\infty$.

Proof. If $m = k - 1$ this lemma follows trivially from Lemma 2.1. When $m < k - 1$ we pick a sequence of elements $\phi_n \in \Phi$ (the unit sphere) each having exactly k zeros, as in Lemma 2.1. The zeros of ϕ_n are the prescribed z_j's together with as many extra zeros as necessary. These extra zeros will, in the limit, all lie in $]-\infty, z] \cap T$ for some z such that $\phi_n(t) = 0$ for all $t \in]-\infty, z] \cap T$. This point z is so chosen that for each n the extra zeros of ϕ_n are in $]-\infty, z] \cap T$ or are elements of a sequence converging to z. The point z is the infimum of the set of points $t \in T$ such that whenever $z_j \leqslant t < z_{j+1}$, $\operatorname{card}]-\infty, t] \cap T \geqslant k - m + j$. Thus, either there are just enough points in $]-\infty, z] \cap T$ for the extra zeros or z is an accumulation point of T from the right.

To this end, define

$$z = \inf\{t \in T \mid z_j \leqslant t < z_{j+1} \Rightarrow \operatorname{card}]-\infty, t] \cap T \geqslant k - m + j \quad \text{for } j = -1, 0,..., m\}.$$

Then $z < +\infty$ since $\operatorname{card} T \geqslant k + 1$, and there exists an i such that $z_i \leqslant z < z_{i+1}$ $(-1 \leqslant i \leqslant m)$. By the definition of z, $\operatorname{card}]-\infty, z] \cap T \leqslant k - m + i$ with strict inequality holding only if z is an accumulation point of T from the right. Define s by card$]-\infty, z] \cap T = s + i + 1$. Then $m + 1 \leqslant k - s$. The $s + i + 1$ points of $]-\infty, z] \cap T$ of course include the $i + 1$ prescribed zeros $z_0, z_1, \dots, z_i$.

We now pick certain points ξ_{jn}, ξ_j satisfying the assumptions for z_{jn} and z_j of Lemma 2.1. The polynomial ϕ given by Lemma 2.1 satisfies the requirements of Lemma 2.2.

Label the elements of $]-\infty, z] \cap T$ in their natural order as ξ_j, $j = 0, 1, \dots, s + i$ and for every $n \in \mathbb{N}$ set $\xi_{jn} = \xi_j$, $j = 0, 1, \dots, s + i$.

If $k - s - (m + 1) > 0$ then z must be an accumulation point of T from the right and since $z < z_{i+1}$ there exist ξ_{jn} for all $n \in \mathbb{N}$, $j = (s + i + 1), \dots, (k - (m + 1) + i)$ such that

$$z < \xi_{jn} < \xi_{(j+1)n} < z_{i+1}$$

and

$$\lim_n \xi_{jn} = z, \qquad j = (s + i + 1), \dots, (k - (m + 1) + i).$$

If $i \leqslant m - 1$ then $k - m + i \leqslant k - 1$. In this case for $j = k - m + i, \dots, k - 1$ define

(a) $\xi_j = \xi_{jn} = z_{j-k+m+1}$ for all $n \in \mathbb{N}$.

It is clear from the definitions of ξ_{jn} and (2.2.2) that

$$\forall n \in \mathbb{N}, \qquad \xi_{(j-1)n} < \xi_{jn}, \qquad j = 0, 1, \dots, k - 1.$$

Hence ξ_{jn}, ξ_j, $j = 0, 1, \dots, k - 1$, $n \in \mathbb{N}$ (with $\xi_{-1} = -\infty$) satisfy all the conditions of Lemma 2.1, whence there exists a ϕ satisfying (2.1.4)–(2.1.6) where z_{jn}, z_j are replaced by ξ_{jn}, ξ_j, respectively. Clearly then (2.2.4) holds, and since for each $j = 0, 1, \dots, m$ there is a p such that $\xi_{pn} = z_j$ for all $n \in \mathbb{N}$, it follows from (2.1.5) that (2.2.5) holds.

Since all the elements of $]-\infty, z] \cap T$ were chosen as zeros of ϕ, condition (2.2.6) is trivially verified if $t \leqslant z$. For $t > z$, $t \in [z_{j-1}, z_j] \cap T$, $j > i$ and hence from (a) above $t \in [\xi_{j-1+k-(m+1)}, \xi_{j+k-(m+1)}]$. Therefore from (2.1.6) (with z_j replaced by ξ_j) $(-1)^{j+k-(m+1)} \phi(t) \geqslant 0$. Hence either $\phi(t)$ or $-\phi(t)$ satisfies (2.2.6) depending on whether $k - (m + 1)$ is even or odd, respectively. Since $\phi(t)$ satisfies (2.2.4), (2.2.5), $(-1)^{k-(m+1)} \phi(t)$ satisfies (2.2.4)–(2.2.6).

(2.3) Lemma. *For some $T \subset \mathbb{R}$, suppose $U \subset \mathscr{C}(T)$ is a T-space of degree k. Let z_j, $j = -1, 0, 1, \dots, m$ (with $z_{-1} = -\infty$) satisfy*

$$z_j \in T, \qquad j = 0, 1, \dots, m. \tag{2.3.1}$$

$$z_{j-1} \leqslant z_j, \qquad j = 1, 2, \dots, m. \tag{2.3.2}$$

$$-1 \leqslant m < k. \tag{2.3.3}$$

Then there exists a $\phi \in U$ *such that*

$$\|\phi\| = 1. \tag{2.3.4}$$

$$\phi(z_j) = 0 \qquad j = 0, 1,..., m. \tag{2.3.5}$$

$$t \in [z_{j-1}, z_j] \cap T \Rightarrow (-1)^j \phi(t) \leqslant 0 \qquad j = 0, 1,..., m+1 \quad \textit{with} \quad z_{m+1} = +\infty. \tag{2.3.6}$$

Proof. Because of the parity of oscillation in (2.3.6), it may as well be assumed that

(a) $z_j < z_{j+2}$, $j = 0, 1,..., m-2$.

In addition we can assume:

(b) $z_j = z_{j+1} < +\infty \Rightarrow]z_{j+1}, z_{j+2}[\cap T \neq \varnothing$, $j = 0,..., m-1$.

Indeed, when $z_j = z_{j+1} < +\infty$ and $]z_{j+1}, z_{j+2}[\cap T = \varnothing$ we can replace $\{z_{-1}, z_0,..., z_m\}$ by $\{z'_{-1}, z'_0,..., z'_{m+1}\}$ such that $z_i' = z_i$ if $i \neq j+1$ and $z'_{j+1} = z_{j+2}$. Then the conclusions of this lemma hold with z_i replaced by z'_i iff they hold with z_i.

With assumptions (a) and (b) we pick certain $z_{jn} \in T$ (for $j = 0, 1,..., m$ and $n \in \mathbb{N}$) such that for each n (with $z_{-1n} = -\infty$) they satisfy the requirements of Lemma 2.2. Whence we get a ϕ_n satisfying (2.2.4)–(2.2.6) for z_{jn}. By appropriately choosing z_{jn} we show that a limit of ϕ_n satisfies (2.3.4)–(2.3.6).

Define, for $j = 0, 1,..., m$, z_{jn} as follows. If $z_j < z_{j+1}$ then let $z_{jn} = z_j$ for all n. If $z_j = z_{j+1}$ then let $\xi_j = \inf]z_j, z_{j+2}[\cap T$. Then $\xi_j \in [z_j, z_{j+2}[$ from (b). If ξ_j is an accumulation point of T then define z_{j+1n} such that $z_{jn} < z_{j+1n} < z_{j+2}$ and z_{j+1n} converges to ξ_j. If ξ_j is not an accumulation point then $\xi_j \in]z_j, z_{j+2}[\cap T$ so pick $z_{j+1n} = \xi_j$ for all n. Notice that in either case $\bigcap_n]z_{jn}, z_{j+1n}[\cap T = \varnothing$ whenever $z_j = z_{j+1}$. Using (a), (b) it is easily shown that such z_{jn} always exists and $z_{jn} < z_{j+1n}$ for $j = 0, 1,..., m$ and $n \in \mathbb{N}$.

For each n, then from Lemma 2.2 we have ϕ_n such that

$$\|\phi_n\| = 1,$$

$$\phi_n(z_{jn}) = 0,$$

$$t \in]z_{j-1n}, z_{jn}[\cap T \Rightarrow (-1)^j \phi(t) \leqslant 0 \quad j = 0, 1,..., m \text{ with } z_{m+1n} = +\infty.$$

Since $\Phi = \{u \in U \mid \|u\| = 1\}$ is compact there is $\phi \in \Phi$ such that for some n_i, $i \in \mathbb{N}$, $\|\phi_{n_i} - \phi\| < 2^{-i}$ since $U \subset \mathscr{B}(T)$. Thus as in the proof of Lemma 2.1 it follows that there is ϕ satisfying (2.3.4)–(2.3.6).

(2.4) THEOREM. *For some* $T \subset \mathbb{R}$ *suppose* $U \subset \mathscr{B}(T)$ *is a T-space of degree k.*

Let z_{jn}, $j = 0, 1, \ldots, m$, $n \in \mathbb{N}$, *and* z_j, $j = -1, 0, 1, \ldots, m$ (*with* $z_{-1} = -\infty$) *satisfy*:

$$\forall n \in \mathbb{N},\ z_{jn} \in T,\ z_j \in \operatorname{cl} T \text{ and } \lim_n z_{jn} = z_j, \quad j = 0, 1, \ldots, m. \tag{2.4.1}$$

$$\forall n \in \mathbb{N},\ z_{(j-1)n} < z_{jn}, \quad j = 1, 2, \ldots, m. \tag{2.4.2}$$

$$-1 \le m \le k. \tag{2.4.3}$$

Then there exists a $\phi \in U$ *such that*

$$\|\phi\| = 1. \tag{2.4.4}$$

$$\exists n_i,\ i \in \mathbb{N} \text{ such that } \lim_i \phi(z_{jn_i}) = 0, \quad j = 0, 1, \ldots, m. \tag{2.4.5}$$

$$t \in\,]z_{j-1}, z_j[\, \cap T \Rightarrow (-1)^j \phi(t) \ge 0; \quad j = 0, 1, \ldots, m+1 \text{ with } z_{m+1} = +\infty. \tag{2.4.6}$$

Proof. From Lemma 2.3, for each $n \in \mathbb{N}$, there exists a $\phi_n \in U$ such that

(1) $\|\phi_n\| = 1$.

(2) $\phi_n(z_{jn}) = 0$, $j = 0, 1, \ldots, m$.

(3) $t \in [z_{j-1n}, z_{jn}] \cap T \Rightarrow (-1)^j \phi(t) \ge 0$, $j = 0, 1, \ldots, (m+1)$ with $z_{-1n} = -\infty$, $z_{(m+1)n} = +\infty$.

Once again using the compactness of Φ and replicating the proof in Lemma 2.1 it follows that a $\phi \in U$ satisfying (2.4.4)–(2.4.6) exists.

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